Course Overview

The subject of the course is "The Dynamical System of Billiards". This course is an "inverted class" in the sense that all lectures are given by students. They will present the material to understand and to prove some theorems on billiards.

The mathematical problem of billiards is to investigate the behavior of a trajectory of a ball on a convex billiard table. The ball moves and its trajectory is defined by the ball's initial position and its initial vector speed. The ball's reflections from the boundary of the table satisfy the condition that the incidence and reflection angles are equal.

Tables with simple shapes, such as circles, ellipses, triangles, rectangles, polygons, etc., already exhibit many interesting behaviors and lead to profound mathematical problems.

The theory of billiards has many applications. We will use several methods to study this topic.

Mathematical Setup

Let C be a strictly convex domain in the plane $\mathbb{R}^2$ bounded by a closed curve of class C^1. The billiard on C is the study of the trajectories of a particle moving without friction within C and bouncing off the boundary ∂C in such a way that the angles of incidence and reflection are the same.

We will start by studying basic examples, where ∂C is a circle or an ellipse. Then, we will show that when C is a general C^2-smooth curve, there are infinitely many periodic orbits. Specifically, a theorem by G. Birkhoff states that for every pair of coprime integers (p, q), there is a periodic orbit of the billiard on C, ordered on C in the same way as the orbit of a point on the circle under rotation by an angle $2\pi\frac{p}{q}$.

Let v be a tangent vector in $\mathbb{R}^2$ of norm 1 at a point on ∂C, and associate to it the pair $(t, \alpha) \in \partial C \times]0, \pi[$, where t is the base point of the vector v and α is the angle it makes with the tangent at t. From a mathematical viewpoint, the billiard is the homeomorphism of the annulus $\partial C \times]0, \pi[$ that associates to $v = (t, \alpha)$ the vector $v' = (t', \alpha')$, where t' is the point of intersection of the trajectory originating from v with ∂C, and α' is the angle of the trajectory at the moment of the bounce.

The map $v \mapsto T(v) = v'$ is the billiard map, and we are interested in its "dynamical" behavior: what can be said about the orbits $\{T^n(v)\}_{n\in\mathbb{Z}}$?

Books

This work will be based on a list of several texts, including the followings:

- Utkir A. Rozikov, *An Introduction to Mathematical Billiards*, World Scientific

- Serge Tabachnikov, *Geometry and Billiards*, Student Mathematical Library, Volume 20, American Mathematical Society

- https://www.dynamical-systems.org/billiard/info.html