MINT Course on Controlled Dynamical Systems: Structured Modelling and Numerical Methods

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Abstract

This document presents the context of structured physical modelling, structure-preseving numerical methods and model order reduction. A description of the course on *Controlled Dynamical Systems: Structured Modelling and Numerical Methods* is provided. A detailed schedule is proposed.

Keywords: port-Hamiltonian systems; partial differential equations; partitioned finite element method; structure-preserving model order reduction

1 Context

The modeling of physical systems based on the representation of intrinsic energy exchanges between different energetic domains allows a modular description of their complex dynamic behaviour. In this context, the port-Hamiltonian framework represents a powerful modeling and control tool. Port-Hamiltonian systems (pHs) link the physical energy of a system and its dynamic behaviour through the definition of a geometric structure, named Dirac structure. This geometric structure, which arises in the modeling step, is instrumental for the stability analysis and the control design. In the case of non-linear or distributed parameter systems, the structure is not only instrumental, but also fundamental to study the solutions in a systematic manner. Even though the formalism is not new [7], it has only been extended to distributed parameter systems in 2002 [17]. Since then, a wide range of physical phenomena involving Partial Differential Equations (PDE) have proved to fit in this formalism, see *e.g.* [15, 1, 4].

In order to stabilize, control or simulate complex multi-physical systems, the port-Hamiltonian framework has to translate from the infinite-dimensional setting (*i.e.* PDE) to the finite-dimensional one (*i.e.* ordinary or more generally Differential Algebraic Equations (DAE)) [16, 2, 13]. Recent works have proved to be very efficient to discretize in a structure-preserving manner many distributed pHs, see *e.g.* [3, 5, 8, 12, 10, 11]. Unfortunately, even though the pHs obtained by discretization is of finite dimension, it can be very large, larger than hundreds of thousands of d.o.f. in most cases. Clearly, this increases the computational burden for control design and prevents the use of real-time observer-based controllers. A prior step to the design of efficient controls is to apply Structure-Preserving Model Order Reduction (SP-MOR), see *e.g.* [14, 6, 9, 13]

2 Description of the course

The course on *Controlled Dynamical Systems: Structured Modelling and Numerical Methods* will deal with distributed parameter systems with boundary control and observation, described as port-Hamiltonian systems (pHs). The focus will be given firstly on physically-structured modelling of open dynamical systems described by PDEs, and secondly on structure-preserving numerical methods, which transform an infinite-dimensional pHs into a finite-dimensional one, mimicking the key power balance at the discrete level: the classical Mixed Finite Element Method (MFEM) is being used in a specific way to achieve this goal. Also an introduction to Reduced Order Modelling (ROM) in a data-driven perspective will be given, making use of the Loewner framework.

The mathematical language will be vector calculus. Examples will be first treated thoroughly in one space dimension, and in a second stage only in higher space dimension. Moreover, a lecture devoted to extensions is planned: it can be devoted either to *exterior calculus and Stokes-Dirac geometrical structures*, or to *finite elements exterior calculus*, or to *symplectic integration to provide discrete time systems.*

In order to be as concrete as possible, the **six lectures** are complemented by **three hands on lab sessions**: the first one will enable to tackle 2D linear problems in Python, making use of GetFEM software, whereas the second one will address 2D nonlinear control problems, together with an example of a coupled heat-wave PDE system; the third one is devoted to model order reduction.

The **Basic Notions** presented throughout the course are the following ones:

1. Conservation laws,
2. Hamiltonian dynamics;
3. Mixed Finite Elements Method,
4. Differential Algebraic Equations,
5. Symplectic numerical schemes;
6. Reduced Order Modelling,
7. Data-driven techniques.

Hence, the content of the whole course is intended to fullfill the requirements of a **Basic Course** of M2RI, for which ISAE-SUPAERO is co-accredited. Also, this course can be taught in 2024-2025 and 2025-2026; however, during fall 2026 another topic intended for engineering students and taught at a basic level will be proposed by teachers of Engineering Schools in Toulouse.

3 A detailed schedule: 26h.

3.1 Introduction to port-Hamiltonian systems (pHs): 3h.

The pHs framework is presented for physically-based control of dynamical systems. It is recalled for lumped parameter systems, and especially the linear case is fully detailed, the link with the classical state space description is given ($A = JQ$, B, $C = B'Q$). The presentation relies on the definition of a Hamiltonian function, the choice of energy variables, the computation of co-energy variables, and the presentation of the dynamical system, as a Differential Algebraic Equation (DAE) putting explicitely the so-called constitutive relations apart. The notion of a Dirac structure is introduced. The usual substitution of the constitutive relations, when possible, enables to write the dynamical system as an Ordinary Differential Equation (ODE). Note the symmetric role of inputs and outputs, the lossless power balance for a dynamical system without damping, or the lossy power balance for a dynamical system with damping. In this latter case, also tackle the introduction of extra dissipative ports to recover a Dirac structure.

3.2 Boundary control of PDEs in one space dimension: 3 h.

The extension to distributed parameter systems is provided through 1D examples first, essentially the wave equation in 1D. The new notion of a Stokes-Dirac structure is presented. The main advantage in the 1D case is that the controls / observations at the boundary are of finite dimension, so it is quite easy and natural to understand and follow the methodoloy with no extra functional analysis complexity.

For the vibrating string, computing the transfer matrix of the multi-input multi-output (MIMO) dynamical system will be done quite easily, in the case of constant coefficients, and in the case of different boundary conditions (force, or velocity); a spectral analysis will be provided. Note that in the case of mixed controls, a pH-DAE or descriptor system with 1 constraint is to be found.

The next example will be the heat equation with the help of a classical quadratic Lyapunov function, resulting in an infinite-dimensional pH-DAE, or descriptor system. Here again the transfer matrix will be computed in the case of different boundary conditions (temperature, or heat flux); a spectral analysis will be provided.

We end this lecture by the 2nd-order differential operator with the Euler-Bernoulli beam, and boundary variables of dimension 4 (instead of 2 in the previous cases).

For wave and beam equations, the stabilization by static output feedback will be presented, and the proof of asymptotic stability of the closed-loop system will be given.

Practical details: this lecture will incorporate some exercises.

3.3 Boundary control of PDEs in higher space dimension: 3h.

Here comes an extra difficulty : the boundary controls / observation live in an infinite-dimensional space; moreover, functional spaces in duality have to be delt with correctly. Note that in this part, vector calculus is being used, we make use of divergence and gradient operators.

This main lecture on pHs in dimension 2 is decomposed as follows:

- The 2D wave equation, with different boundary controls: velocity or normal stress, lossless energy balance, available pairs of collocated boundary controls/observations
- The 2D heat equation with quadratic Lyapunov functional, towards a pH-DAE, , lossy energy balance, available pairs of collocated boundary controls/observations
- Extensions: the models are presented to show the effectiveness of the approach in curvilinear coordinates, and with other differential operators: Shallow Water Equation (SWE) in 2D in polar coordinates, *curl* operator for Maxwell's equation, *Div* and *Grad* tensorial operators for the Reissner-Mindlin and Kirchhoff-Love plate equations.

Practical details: this lecture will incorporate some exercises.

3.4 Structure-preserving FEM method for pHs: 3h.

This part will provide theory, numerics through the Partitioned Finite Element Method (PFEM) and worked-out examples on the following models: waves, Euler-Bernoulli beam and heat equation. Moreover, in order to remain self-contained, a short reminder on the Finite Element Method (FEM) will be provided at this stage on the special case of systems in one space variable: Lagrange and Hermite polynomial bases.

- The 2D wave equation, with different boundary controls: velocity or normal stress, application of PFEM to get an ODE. In the case with mixed controls, a pH-DAE or descriptor system with 1 constraint is found.
- The 2D heat equation with quadratic Lyapunov functional, application of PFEM to get a full DAE.
- Discussion on the appropriate choice of conforming finite elements (Lagrange, Raviart-Thomas, Nédélec...)
- Discussion on numerical linear algebra: ODE with full matrices, versus DAE with sparse matrices.
- Extensions: the models are presented to show the effectiveness of PFEM in curvilinear coordinates, and with other differential operators: Shallow Water Equation (SWE) in 2D in polar coordinates, *curl* operator for Maxwell's equation, *Div* and *Grad* tensorial operators for the Mindlin and Kirchhoff plate equations.

Practical details: this lecture will incorporate some exercises.

3.5 Lab-1: 3h.

A first Lab, using GetFEM[1] and SCRIMP[2] as classical programming tools in Python for the applied mathematics community. The examples presented in the lab session will be treated thoroughly, with e.g. comparisons of transfer functions and spectra in the 1D case. For the 2D case, a time-domain simulation of the linear vibrating membrane will be perfomed, and implementation of control laws will be studied, such as the stabilization by an appropriate static output feedback.

Practical details: GetFEM, SCRIMP, Python.

3.6 Possible Extensions: 3h.

The theme of this lecture can be discussed with the students, or changed from one year to the other, only one of them will be treated. Three main topics can be proposed:

1. Exterior differential calculus using differential forms is an elegant mathematical framework to describe in a coordinate-free setting many physical phenomena, e.g. (finite-dimensional) Hamiltonian dynamics of mechanical systems or (infinite-dimensional) systems of conservation or balance laws like the Maxwell equations of electrodynamics. In contrast to vector calculus, a single differential operator, the exterior derivative, is defined in which terms the operations grad, rot, div can be expressed. This requires an additional operation, the Hodge star, which relates the canonical duality pairing of differential forms and the standard (L_2) inner product on an infinite-dimensional space. The sequence of spaces of differential forms, which are related by the exterior derivative, is the so-called de Rham complex. Functional spaces of differential forms in geometric discretization shall form appropriate sub-sequences of this chain complex.

2. Finite Element Exterior Calculus (FEEC) which uses the concept of Finite Elements in a more abstract way, and proves fully compatible with the exterior calculus theory. In particular, the definition and generation of classes of finite elements is well understood, and even applied in the FEniCS library[3]. One particular point of interest is the preservation of the de Rahm cohomology at the discrete level.

3. Symplectic numerical schemes are also appropriate candidates for the discretization of (open) PH systems, and we will present a definition of discrete-time PH systems based on symplectic integration. These schemes

[1] `https://getfem.org/`
[2] `https://g-haine.github.io/scrimp/`
[3] `https://fenicsproject.org/`

are very important in order to mimick the energy balance of the semi-discrete approximation at the fully discrete level. The most popular such schemes are: Euler A, Euler B, Störmer-Verlet, inverse Störmer-Verlet (the four of them are explicit, 1st- or 2nd-order accurate), while the classical Crank-Nicholson scheme is also 2nd-order accurate but implicit.

3.7 Lab-2: 3h.

For the 2D case, a time-domain simulation of the non-linear shallow water equations (SWE) will be perfomed, and implementation of control laws will be studied, such as the stabilization around an equilibrium water height by an appropriate static output feedback. The discretized heat equation will be studied as a pH-DAE. The objective of this second lab is the interconnection between a wave and a heat equation, in a structure-preserving way, which might allow the preservation of refined asymptotics behaviours of the coupled system (polynomial decay, logarithmic decay), see [10].

Practical details: GetFEM, SCRIMP, Python.

3.8 Structure-preserving Reduced Order Model: 2h.

Computing simplified, easy to use dynamical models is one purpose of the model approximation and reduction discipline. The goal is to approximate the original system with a smaller and simpler system with the same structure and similar response characteristics as the original, the low- complexity model, also called a reduced order model (ROM). The Loewner framework (LF) is a data-driven model identification and reduction technique that was introduced recently. Using only frequency- domain measured data, the LF constructs surrogate models directly and with low computational effort. Its extension to pHs model was proposed a few years ago, and successful attempts to apply data-driven techniques to identification of pHs have emerged since then, on 1D examples and also on 2D examples.

3.9 Lab-3: 3h.

In this final lab, some high-fidelity models (HFM) provided by PFEM in Lab 1 or in Lab 2 will be reduced to low-fidelity models (LFM), while keeping the port-Hamiltonian structure of the reduced system. To this end, the MOR Matlab toolbox[4] will be used.

Practical details: Matlab, MOR Toolbox.

[4] `https://mordigitalsystems.fr/en/products.html`

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