Syllabus for the M2 course

Uncertainty principles and null-controllability

Paul Alphonse

In the first part of this course, we will present some uncertainty principles related to Heisenberg's famous uncertainty principle, a mathematical version of which roughly states that a function in $L^2(\mathbb{R}^d)$ cannot be too localized in both space and frequency. More precisely, given two measurable sets $\Sigma, S \subset \mathbb{R}^d$, we aim to understand under what conditions on the pair (S, Σ) the following property is satisfied

$$\forall f \in L^2(\mathbb{R}^d), \quad (\operatorname{supp} f \subset S \quad \text{and} \quad \operatorname{supp} \widehat{f} \subset \Sigma) \quad \Longrightarrow \quad f = 0.$$

We will also be interested in pairs (S, Σ) that satisfy the following stronger property

$$\exists C(S,\Sigma) > 0, \forall f \in L^2(\mathbb{R}^d), \quad \|f\|_{L^2(\mathbb{R}^d)} \leq C(S,\Sigma)\big(\|f\|_{L^2(\mathbb{R}^d \setminus S)} + \|\widehat{f}\|_{L^2(\mathbb{R}^d \setminus \Sigma)}\big). \tag{1}$$

We will begin by stating and proving some qualitative results, including the Logvinenko-Sereda theorem, which gives a complete characterization of the pairs (S, Σ) satisfying condition (1) when Σ is assumed to be bounded (in addition to being measurable). We will then present a precise quantitative uncertainty principle for a class of analytic functions, whose proof is based on an approach due to Kovrijkine, and which notably allows us to give an expression for the constant $C(S, \Sigma)$ in the inequality (1) in the context of the aforementioned Logvinenko-Sereda theorem.

The second part of this course aims to apply the previously established uncertainty principles to the study of the null-controllability of the heat equation posed on $\mathbb{R}^d$

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = h(t,x)\mathbb{1}_\omega(x), & t > 0, \ x \in \mathbb{R}^d, \\ u(0,\cdot) = u_0 \in L^2(\mathbb{R}^d), \end{cases} \tag{2}$$

where $\omega \subset \mathbb{R}^d$ is a measurable set with positive measure. Given a time $T > 0$ and an approximation parameter $\varepsilon \geq 0$, we aim to determine the correct geometry to impose on the support $\omega \subset \mathbb{R}^d$ in order to null-control the equation (2) exactly or approximately. In other words, for any initial data $u_0 \in L^2(\mathbb{R}^d)$, we seek to determine whether it is possible to find a control h localised in $[0,T] \times \omega$ such that

$$\|u(T,\cdot)\|_{L^2(\mathbb{R}^d)} \leq \varepsilon \|u_0\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|h\|_{L^2([0,T]\times\omega)} \leq C_{\varepsilon,\omega,T}\|u_0\|_{L^2(\mathbb{R}^d)}.$$

The case $\varepsilon = 0$ corresponds to the exact control and the case $\varepsilon > 0$ to the approximate control. The link between the concepts of uncertainty principle and null-controllability is made via the Hilbert Uniqueness Method, which states that the concepts of exact and approximate null-controllability are respectively equivalent to the concepts of observability and approximate observability, which are written as

$$\big\|e^{T\Delta}u_0\big\|^2_{L^2(\mathbb{R}^d)} \leq C_{\varepsilon,\omega,T} \int_0^T \big\|e^{t\Delta}u_0\big\|^2_{L^2(\omega)}\,\mathrm{d}t + \varepsilon\|u_0\|^2_{L^2(\mathbb{R}^d)},$$

the case $\varepsilon = 0$ corresponding to the exact observability and the case $\varepsilon > 0$ to the approximate observability. We will present in detail the Hilbert Uniqueness Method, as well as Miller's telescopic series method, which allows us to obtain precise exact observability costs $C_{0,\omega,T}$. Depending on the progress of the course, other topics related to null-controllability may also be covered.

Possible plan of the course

CHAPTER 0 - Remainders on the Fourier transform on $L^2(\mathbb{R}^d)$.

CHAPTER 1 - Qualitative uncertainty principles.

CHAPTER 2 - Quantitative uncertainty principles.

CHAPTER 3 - Null-controllability of the heat equation on $\mathbb{R}^d$.

Some references

1. V. HAVIN & B. JÖRICKE, *The uncertainty principle in harmonic analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 28, Springer-Verlag, Berlin (1994).
2. O. KOVRIJKINE, *Some results related to the Logvinenko-Sereda Theorem*, Proc. Amer. Math. Soc. **129**, (2001), 3037–3047.
3. L. MILLER, *A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups*, Discrete Contin. Dyn. Syst. Ser. B **14** (2010), 1465–1485.