Introduction to Optimal Mass Transport

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Abstract:

This course is meant to be an introduction to optimal mass transport. Optimal Mass Transport deals with a variational problem involving two (or more) probability measures (being the distributions of mass we consider). Given a cost function modeling the cost of moving a unit of mass from one place to another, and a transport plan (a way to move one distribution of mass onto the other), one can define the average cost of the transport plan. The general question is then to seek for minimisers of this variational problem and the features of the solutions.

The optimal mass transport problem admits plenty of variants depending on the choice of the cost function, the space on which the problem is defined -this space can be a general Polish metric space or a Riemannian manifold or simply the Euclidean space for instance-, and the type of solutions studied (Monge's solutions or Kantorovitch's ones).

This problem can be used to prove functional inequalities, to define a meaningful distance metricizing the weak convergence of probability measures called Wasserstein metric. It also provides weak solutions to pdes of Monge-Ampère type.

We shall also discuss applications to convex geometry by considering a non-standard cost function on the unit sphere in Euclidean space.

We also plan to introduce the notion of barycenters of measures which provides a useful notion of mean of probability measures on non-flat spaces and is useful in statistics for instance.

More specifically, the following topics will be adressed in the course:

- General introduction to the problem of mass transportation on a metric space.
- Kantorovitch's duality
- Monge's problem and Brenier's theorem in Euclidean space.
- Wasserstein spaces and weak convergence of probability measures on a Polish space.

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- Displacement interpolation.
- Multi-marginal problem and barycenter of measures.
- (Optional) Needle decomposition in Euclidean space.
- Applications to be discussed during the course:
  - Proof of the isoperimetric inequality/Sobolev inequality,
  - Inverse problems in convex geometry and non-standard cost functions,
  - Variations on the Wasserstein metric and applications in computer vision,
  - Other barycenters and the notion of mean in non-flat statistics.
  - Behavior of Entropy and other functionals in the Wasserstein space.

References:

- L. Ambrosio, E. Brué, D. Semola *Lectures on Optimal Transport*, Springer, 2021.
- F. Santambrogio *Optimal transport for applied mathematicians*, Birkhäuser, 2015.
- C. Vilani *Topics in Optimal Transportation*, AMS, 2003.
- C. Villani, *Optimal transport Old and new*, Springer, 2008.