Course A6 : An introduction to the theoretical and numerical analysis of scalar conservation laws

F. Boyer, franck.boyer@math.univ-toulouse.fr

The goal of this course is to introduce the audience to the study of partial differential equations of hyperbolic type and more precisely of scalar conservation laws of the form

$$\partial_t \rho + \mathrm{div}_x \big(F(t, x, \rho) \big) = 0,$$

where $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a given vector field. We will particularly concentrate on the linear case where $F(t, x, \rho) = \rho\, v(t, x)$ with $v : \mathbb{R} \times \Omega \to \mathbb{R}^d$ and on the one-dimensional nonlinear case where $F(t, x, \rho) = f(\rho)$ with $f : \mathbb{R} \to \mathbb{R}$.

Prerequisites : Differential calculus and differential equations. Basic functional analysis. Lebesgue integration theory.

1. Modelling of transport phenomena
   - Some examples in the 1D case (traffic flow, gas dynamics, ...)
   - The multi-D case : Reynolds transport theorem. Continuity and transport equations.
2. The linear transport equation
   - Smooth solutions by the method of characteristics.
   - Finite difference schemes in 1D : construction, consistency, stability, convergence, error estimates. Monotonicity.
   - Weak solutions of the linear transport equation. Jump conditions. Existence and uniqueness.
   - Finite volume methods in 1D : construction, stability, convergence towards weak solutions.
   - Extensions to the multi-D setting on arbitrary grids. Weak BV estimates.
3. Nonlinear conservation laws in 1D
   - Using the characteristics : local in time well-posedness of smooth solutions, finite time singularities, nonlinear regularization process.
   - Study of weak solutions : Rankine-Hugoniot conditions, global in time existence, non uniqueness.
   - Entropy solutions : physical motivations, definition and characterizations (Lax conditions, Oleinik conditions).
   - Riemann problems : definition, shocks and rarefaction waves.
   - Krushkov entropies.
   - Finite volume methods : monotone fluxes, TVD schemes. Convergence towards an entropy solution for BV data.
   - Uniqueness of the entropy solution by the doubling variables technique. Krushkov theorem.
4. Possible extensions
   - Boundary conditions.
   - Finite volume schemes for scalar conservation laws in any dimension.
   - Few words concerning hyperbolic systems.

References

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[4] A. Harten, J. M. Hyman, and P. D. Lax. On finite-difference approximations and entropy conditions for shocks. *Comm. Pure Appl. Math.*, 29(3):297–322, 1976. With an appendix by B. Keyfitz.

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