Convergence of Probability Measures and Introduction to Optimal Transport

Overview

This lecture is divided into two main parts:

- **Part 1: Convergence of Probability Measures**
- **Part 2: Introduction to Optimal Transport**

Part 1: Convergence of Probability Measures

I. Introduction and Preliminary Concepts

- **Basic Definitions:** Recall of stochastic convergence (e.g., convergence in distribution, almost sure convergence, tight convergence).
- **Motivation:**
  - Central Limit Theorem (CLT)
  - Donsker Invariance Principle: The Brownian motion as the limit of properly scaled random walks.

II. Topology of Convergence in $\mathcal{M}_1(E)$ (Set of Probability Measures)

- **Polish Space:** Complete, separable metric spaces.
- **Tightness and Prokhorov Theorem:**
  - Definition of tightness and its role in convergence.
  - Prokhorov's theorem ensures compactness in $\mathcal{M}_1(E)$.
- **Properties of the Topology in $\mathcal{M}_1(E)$:** Case of $\mathbb{R}$ and $\mathbb{R}^d$ and general case.
- **Metrics on $\mathcal{M}_1(E)$:** Example of the Lévy-Prokhorov metric...

III. Functional Limit Theorems

- **Donsker Invariance Principle:** Detailed proof using convergence of stochastic processes.
- **Topology of the continuous Skorokhod Space:**
- **Kolmogorov Criterion:** Sufficient conditions for tightness in the Skorokhod space.

Part 2: Introduction to Optimal Transport

IV. Monge-Kantorovich Problem

- **Monge's Problem:** Historical context, definitions, and examples.
- **Kantorovich's Relaxation:** Reformulation as a convex optimization problem.

V. Existence of Optimal Transport Plans

- Proofs and examples of existence under specific conditions.

VI. Optimal Transport in One Dimension

- **Discrete Case:** Transport between two discrete distributions.
- **General Case:** Properties in the continuous setting.

VII. Optimal Transport in Higher Dimensions

- **Cyclic Monotonicity:** Key property in higher dimensions.
- **Brenier's Theorem:** Existence and uniqueness for quadratic cost.

VIII. Wasserstein Distance

- Definition and properties of the Wasserstein metrics W_p.
- Connections to probability measure convergence and applications.

Reference

1) P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley Series in Probability and Statistics, John Wiley & Sons, New York, 1999.

2) G. Peyré and M. Cuturi, Computational Optimal Transport, ArXiv:1803.00567, 2018.

3) C. Villani, *Optimal Transport: Old and New*, Grundlehren der mathematischen Wissenschaften, vol. 338, Springer-Verlag, Berlin, Heidelberg, 2009.