Affine Surfaces, Homogeneous Vector Fields and Germs Tangent to the Identity

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This course will complement the course on hyperbolic and translation surfaces. We will show how the study of the long term behavior of geodesics (lines) in translation or affine surfaces may be used to understand the real-time dynamics of homogeneous vector fields in $\mathbb{C}^2$ or the dynamics of germs tangent to the identity in $\mathbb{C}^2$.

A translation surfaces (respectively an affine surface), is a surface one obtains by gluing together polygons via translation (respectively via complex affine map $z \mapsto \lambda z + \mu$), for example a cube, a cylinder or a cone in $\mathbb{R}^3$. Geodesics in such a surface are parameterized euclidean lines.

Studying the dynamics of a homogeneous vector field on $\mathbb{C}^2$ amounts to studying the solutions of a differential equation

$$\begin{cases} x' = P(x, y) \\ y' = Q(x, y) \end{cases}$$

where $P : \mathbb{C}^2 \to \mathbb{C}$ and $Q : \mathbb{C}^2 \to \mathbb{C}$ are homogeneous polynomials.

Discrete holomorphic dynamics in complex dimension 2 is the study of sequences defined by induction

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in \mathbb{C}^2, \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = F \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

where $F : \mathbb{C}^2 \to \mathbb{C}^2$ is a holomorphic map.

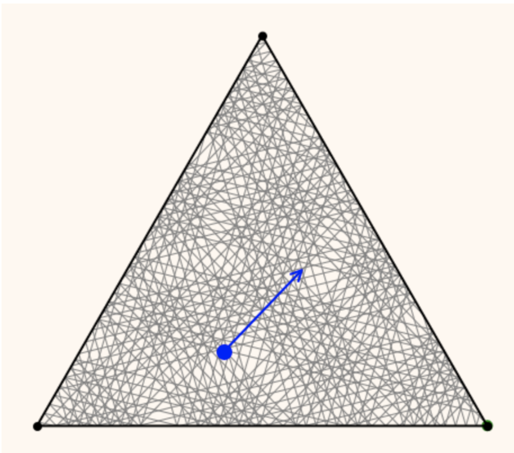
We will see that there is a rich interaction between those three domains. In particular, we will see that studying sequences of the form

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} y_n^2 \\ x_n^2 \end{pmatrix}$$

relies on the study of the differential equation

$$\begin{cases} x' = y^2 \\ y' = x^2 \end{cases}$$

which itself may be understood by studying the geodesics in the affine surface obtained by gluing two equilateral triangles along their edges. This will lead us to the study of trajectories in triangular billiard tables.



Prerequisites

Basic knowledge on holomorphic functions. Attendance of the course An introduction to hyperbolic and translation surfaces is strongly required.

References

- A. Zorich, Flat surfaces, https://arxiv.org/abs/math/0609392
- M. Abate, Fatou flowers and parabolic curves, https://arxiv.org/abs/1501.02176