REGULARITY THEORY
FOR MINIMIZING HARMONIC MAPS

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Abstract. Minimizing harmonic maps are natural higher dimensional generalizations of geodesics. In contrast to geodesics however, harmonic maps have singularities, due to topological obstructions or energy efficiency.

More specifically, we consider maps u from an open set in $\mathbb{R}^d$ with values into a closed subset $\mathcal{N}$ in $\mathbb{R}^n$. We then define the energy

$$\mathcal{E}(u) = \int_\Omega |Du(x)|^2 \, dx.$$

We thus assume implicitly that each coordinate u_j of u belongs to the Sobolev space $W^{1,2}(\Omega)$ for $j = 1, \dots, n$, so that the integrand $|Du(x)|^2 = |Du_1(x)|^2 + \cdots + |Du_n(x)|^2$ is summable on Ω.

We study the so-called *minimizing harmonic maps* $u \in W^{1,2}(\Omega; \mathcal{N})$ which minimize the energy $\mathcal{E}$, in the sense that for every competitor w agreeing with u on $\partial\Omega$, one has

$$\mathcal{E}(u) \leq \mathcal{E}(w).$$

When $\mathcal{N}$ is a compact submanifold of $\mathbb{R}^n$ without boundary, a classical theorem due to Schoen and Uhlenbeck asserts that such minimizers are smooth outside a singular set of dimension less than $d-3$. This result is optimal in view of the classical example $x \mapsto \frac{x}{|x|}$ from the ball $\mathbb{B}^3$ into the sphere $\mathbb{S}^2$.

This series of three lectures is an essentially self-contained introduction to minimizing harmonic maps, with special emphasis on the proof of the above regularity theorem. The latter involves several important tools, which play a crucial role in many fields of Analysis, including the monotonicity formula, reverse Poincaré inequalities, compactness theorems and density functions.

References

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