Syllabus – Advanced course
Branching Brownian motion and variants

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Overview. Branching Brownian motion (BBM) is a particle system on the real line defined as follows. It starts at time 0 with one particle at the origin. The particles evolve independently in the following way: each particle moves according to a Brownian motion during an exponentially distributed lifetime, at which it dies and is replaced by a random number L of new particles, where L follows some given distribution, called the *offspring distribution*. This simple model naturally arises in various contexts, such as biology (population models), computer science (analysis of algorithms), statistical physics (spin glasses, random matrices, Coulomb gases, quantum gravity), analytic number theory (Riemann zeta function), and partial differential equations (reaction-diffusion equations).

In this course, we describe the evolution of this particle system as time goes to infinity, with for example results about the growth of the population in different regions of the real line or about the asymptotic position of the maximal particle. We also study variants of the model which include heterogeneity or selection effects.

This course brings to action many probabilistic tools, such as: second moment method, martingales and stochastic calculus. It also provides a glimpse into an extremely active research area.

Detailed content

- **Introduction.** Definition of the BBM. Presentation of the basic tools: many-to-one and many-to-two lemmas.
- **The bulk.** Definition and study of the so-called *additive martingales*, with the use of L^p techniques. Application to the growth in the bulk. Link with reaction-diffusion PDEs.
- **The maximum.** Proof of the first and second order for the position of the highest particle at time t. Comparison with the case of independent displacements to emphasize the role of the hierarchical structure. Proof of the convergence in law of the recentered maximum. For this, definition and study of the *derivative martingale*.
- **Time-inhomogeneous BBM.** Study of a BBM where variance of the displacement changes over time. Behavior of the maximum for these models, to see the strong difference between the case of increasing and decreasing variance.
- **Model with selection.** Study of the BBM with killing along a line. Phase transition depending on the slope of the line. Asymptotics of the survival probability in the near-critical case.