M2RI – Reading Seminar 2

Part I

Unique continuation for elliptic partial differential equations

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Part II

Introduction to reaction-diffusion equations

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Prerequisite: We recommend students to have taken the course A5 on *Sobolev spaces & Elliptic equations*.

Part I: Unique continuation for elliptic partial differential equations

This reading seminar is dedicated to the study of *Unique Continuation Properties* (UCP) for second order elliptic partial differential equations. A typical and well-known example of such UCP is the following: let Ω be a non-empty connected open domain of $\mathbb{R}^d$, and ω be a non-empty open subset of Ω. Then any harmonic function u that is identically zero in ω is identically zero in Ω. Schematically:

$$\begin{cases} \Delta u = 0 \text{ in } \Omega, \\ u = 0 \text{ in } \ \omega, \end{cases} \quad \Rightarrow \quad u = 0 \text{ in } \Omega.$$

We will demonstrate that result, discuss some extensions, in particular the so-called *propagation of smallness* which is a quantified version of the UCP, and some applications of UCP to inverse problems and spectral theory.

References

[1] J. Dardé and S. Ervedoza. A short introduction to Carleman estimates. Control of partial differential equations. *Ser. Contemp. Appl. Math. CAM,* 24, 2023. HAL version.

[2] M. V. Klibanov and A.A. Timonov. Carleman estimates for coefficient inverse problems and numerical applications. *Inverse and Ill-Posed Problems Series*, (2004).

[3] J. Le Rousseau, G. Lebeau and L. Robbiano. Elliptic Carleman estimates and applications to stabilization and controllability. Volume I. Dirichlet boundary conditions on Euclidean space. *Progress in Nonlinear Differential Equations and Their Applications* 97, Subseries in Control, 2022.

[4] S. Vessella. Notes on unique continuation properties for Partial Differential Equations – Introduction to the stability estimates for inverse problems. *Preprint arXiv*, 2023.

Part II: Introduction to reaction-diffusion equations

In this seminar, we introduce some basic tools for the study of reaction-diffusion equations which belong to the class of nonlinear parabolic PDEs. We will focus on one specific reaction-diffusion equation, namely the scalar Fisher-KPP (1937) equation for which we will present a self-content analysis of the long time dynamics of its solutions. Such an equation arises not only in the present context, but also in the study of the Brownian motion (see Advanced Course B5 on *Branching Brownian motion and variants*). More specifically, there will be three different parts in this reading seminar.

- **Part 1:** We will start by an introduction of the equations and the models under study, and explain some motivations coming from other disciplines such as spatial ecology, evolution, physics, chemistry and epidemiology.
- **Part 2:** We will review (and/or introduce) some basic techniques for the study of parabolic equations such as maximum principles, notions of sub/super-solutions, regularity theory, etc... We will also deal with the existence, uniqueness and regularity of the solutions of the Cauchy problem.
- **Part 3:** We will study the long time behavior of the solutions of the Cauchy problem starting from compactly supported initial data, and characterize so-called spreading properties of the solutions.

References

[1] D.G. Aronson and H.F. Weinberger. Multidimensional nonlinear diffusions arising in population genetics. *Adv. Math.* 30, pp. 33-76, 1978.

[2] R.A. Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7, pp. 353-369, 1937.

[3] A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov. Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. Etat Moscou, Ser. Inter.* A 1, pp. 1-26, 1937.

[4] M. H. Protter and H.F. Weinberge. Maximum principles in differential equations. *Springer Science & Business Media* (2012).