M2RI – Reading Seminar 2

Part I

Scaling limits in statistical mechanics

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Part II

Introduction to reaction-diffusion equations

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Part I: Scaling limits in statistical mechanics

Introduction

Statistical mechanics studies the properties of large particle ensembles where particles interact with each other according to the laws of mechanics. Since the systems are typically very large (for example, in a gas there are around 10^{23} molecules in a unit of volume), the derivation of reduced models describing the statistics of the system, such as the so-called **kinetic models**, has became a central problem in the field.

In this seminar we will study some mathematical methods to obtain kinetic models through **many-particle limits**, such as the mean-field and the Boltzmann-Grad limits. In particular, we will outline the main steps of the derivation of some classical kinetic equations, namely

- the Vlasov equation (1938), used in the modelling of plasmas,
- the Smoluchowski equation (1916) for polymerization and aerosol dynamics
- the Boltzmann equation (1872) for rarefied gases
- or the Hartree equation (1927) used in quamtum chemistry.

Possible topics and references

We will study between 2 to 4 topics chosen among the six topics listed below.

- **Mean-field limits**

  In the case of systems in which the behaviour of each particle is affected by all the other particles, the mean-field limit allows to replace the full system by one effective equation for one typical particle under the effect of an average interaction. Possible topics on the derivation of kinetic equations using mean-field limits are:

  1. From Newton to Vlasov equation: mean-field limit in classical mechanics ([4], Lecture 1)
  2. From Schrödinger to Hartree equation: mean-field limit in quantum mechanics ([4], Lecture 3)

- **Pairwise interactions and the Boltzmann-Grad limit**

  If instead the particles only interact locally, for example, in the case of pairwise collisions in rarefied gas, then the mean-field theory is not suitable. Alternative approaches include for example the so-called Boltzmann-Grad limit. Possible topics and references in this direction are:

  3. From the deterministic hard-sphere system to the Boltzmann equation ([1] and [2], Section 4.6)
  4. From coagulating Brownian particles to the Smoluchowski coagulation-diffusion equation ([5], Section 4)

- **Pairwise interactions in spatially homogeneous models**

  Other scalings can be used to obtain kinetic modes in the many-particles limit.

  5. From the Kac's model to the Boltzmann equation ([2], Section 4.1)
  6. From coalescing particles to the Smoluchowski coagulation equation ([6], Section 4)

Prerequisites

Functional analysis or stochastic processes.

References

[1] C. Saffirio. Derivation of the Boltzmann equation: hard spheres, short-range potentials and beyond. In: From Particle Systems to Partial Differential Equations III: Particle Systems and PDEs III, Braga, Portugal, (2016) 301–321. Springer International Publishing.

[2] L. P. Chaintron, and A. Diez. Propagation of chaos: A review of models, methods and applications II. *Kinetic and Related Models* 15.6 (2022): 1017-1173.

[3] M. Escobedo, F. Pezzotti. Propagation of chaos in a coagulation model. Mathematical Models and Methods in Applied Sciences, 23(06) (2013) 1143–1176.

[4] F. Golse, Mean-field limits in statistical dynamics. *arXiv preprint* (2022) arXiv:2201.02005.

[5] R. Lang and N. X. Xanh. Smoluchowski's theory of coagulation in colloids holds rigorously in the Boltzmann-Grad-limit. *Z. Wahrscheinlichkeitstheor. verw. Geb.*, **54** (3) (1980) 227-280.

[6] J. R. Norris, Smoluchowski's coagulation equation: Uniqueness, nonuniqueness and a hydrodynamic limit for the stochastic coalescent. *Ann. App. Probab.* (1999) 78-109.

Part II: Introduction to reaction-diffusion equations

In this seminar, we introduce some basic tools for the study of reaction-diffusion equations which belong to the class of nonlinear parabolic PDEs. We will focus on one specific reaction-diffusion equation, namely the scalar Fisher-KPP (1937) equation for which we will present a self-content analysis of the long time dynamics of its solutions. Such an equation arises not only in the present context, but also in the study of the Brownian motion (see Advanced Course B5 on *Branching Brownian motion and variants*). More specifically, there will be three different parts in this reading seminar.

- **Part 1:** We will start by an introduction of the equations and the models under study, and explain some motivations coming from other disciplines such as spatial ecology, evolution, physics, chemistry and epidemiology.
- **Part 2:** We will review (and/or introduce) some basic techniques for the study of parabolic equations such as maximum principles, notions of sub/super-solutions, regularity theory, etc... We will also deal with the existence, uniqueness and regularity of the solutions of the Cauchy problem.
- **Part 3:** We will study the long time behavior of the solutions of the Cauchy problem starting from compactly supported initial data, and characterize so-called spreading properties of the solutions.

Prerequisite: We recommend students to have taken the course A5 on *Introduction to PDEs*.

References

[1] D.G. Aronson and H.F. Weinberger. Multidimensional nonlinear diffusions arising in population genetics. *Adv. Math.* 30, pp. 33-76, 1978.

[2] R.A. Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7, pp. 353-369, 1937.

[3] A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov. Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. Etat Moscou, Ser. Inter.* A 1, pp. 1-26, 1937.

[4] M. H. Protter and H.F. Weinberge. Maximum principles in differential equations. *Springer Science & Business Media* (2012).