9) Bonus 2 ⓐ $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ donc $|\cos z|^2 = \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{-i\bar z}+e^{i\bar z}}{2}\right)$

$z = x+iy$,

$|\cos z|^2 = (e^{-2y} + e^{2y} + e^{2ix} + e^{-2ix})/4$

$= (sh(y))^2 + (\cos x)^2$

$sh(y) = \frac{e^y - e^{-y}}{2}$

De même, $|\sin z|^2 = |sh(y)|^2 + (\sin x)^2$

Donc $|\cotan(\pi z)|^2 = \frac{|\cos(\pi z)|^2}{|\sin(\pi z)|^2} = \frac{(sh(\pi y))^2 + (\cos(\pi x))^2}{(sh(\pi y))^2 + (\sin(\pi x))^2}$

$\leq \frac{1+(sh(\pi y))^2}{(sh(\pi y))^2} = |\cotan h(\pi y)|^2$

donc $|\cotan(\pi z)| \leq \cotanh|\pi y| \leq \cotanh\left(\frac{\pi}{2}\right) = \frac{1+e^{-\pi}}{1-e^{-\pi}}$

pour $|y| \geq \frac{1}{2}$.

ⓑ Pour $|y| \leq 1/2$

$\left|\cotan\left(\pi(k+\tfrac{1}{2}+iy)\right)\right| = |\tan(\pi i y)| = |\tanh(\pi y)| \leq \tanh\left(\frac{\pi}{2}\right)$

$= \frac{1-e^{-\pi}}{1+e^{-\pi}}$

ⓒ Soit $M \geq \sup\left(\frac{1+e^{-\pi}}{1-e^{-\pi}}, \frac{1-e^{-\pi}}{1+e^{-\pi}}\right)$.