Project M2 RI : First passage percolation on random graphs

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The most basic and classical model of a random graph is the *Erdős–Rényi graph* $\mathcal{G}_{n,p}$. Its vertex set is $[n] = \{1, 2, \ldots, n\}$ and for every $i, j \in [n]$, $i \neq j$, an edge between i and j is present in the graph with probability p, independently for all pairs $\{i, j\}$. We write $i \sim j$ if i and j are connected by an edge. This graph has intensely been studied in the last 60 years. In particular, when $n \to \infty$, if $p \geq \lambda/n$ for some $\lambda > 1$, it is known that it has a *giant component* with high probability, i.e. a component that contains a non-vanishing proportion of vertices.

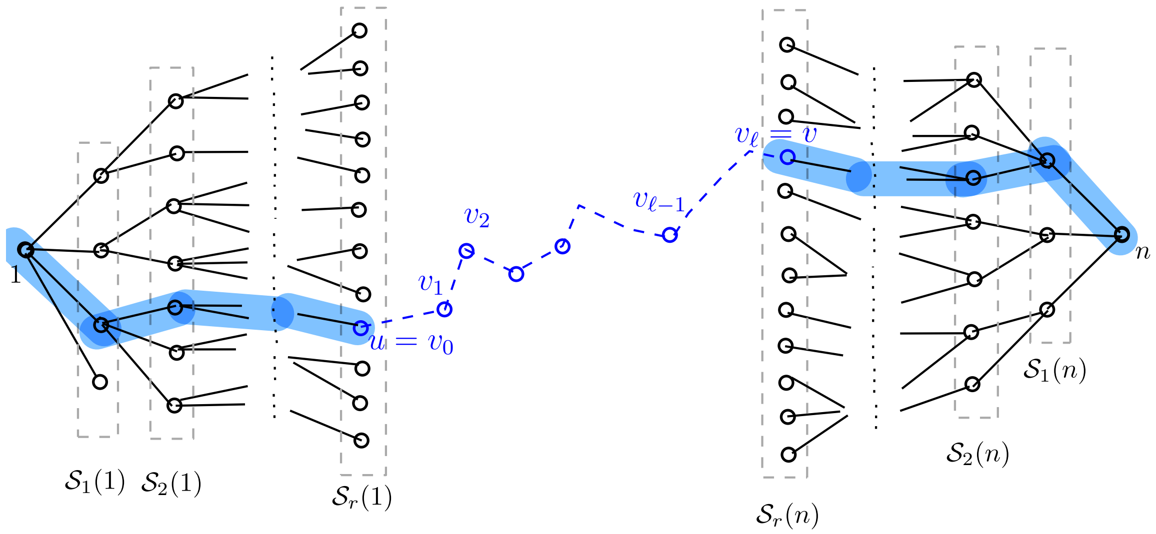

Figure 1: Schematic description of the geometry of the Erdős–Rényi graph. The local neighborhood of vertices 1 and n are tree-like and both are connected by long paths in between. The first passage percolation distances in the local neighborhoods can be coupled to branching random walks.

Now, take such a random graph $\mathcal{G}_{n,\lambda/n}$ with $\lambda > 1$, and assign iid random variables $X(e)$ to every edge in the graph, which follow the same law as some random variable X. Given a path $p = (v_0, \ldots, v_k)$, with $v_{i-1} \sim v_i$ for every i, we define its *weight* to be the sum of the weights along the edges:

$$X(p) = \sum_{i=1}^{k} X(\{v_{i-1}, v_i\})$$

Now fix two vertices, say 1 and n. They are both in the giant component with probability bounded away from 0, in which case they are connected by a path (and possibly many). We are interested in the path of *minimal weight* between these two vertices. To avoid ambiguities when the edge weights can be negative, we only consider *simple paths*, in which every vertex is visited at most once. We therefore define

$$\mathbb{X}_n = \min_{p\text{: simple path from 1 to } n} X(p).$$

In [1], under certain assumptions on the law of X which guarantee that $\mathbb{X}_n$ is positive with high probability, we prove that $\mathbb{X}_n$, suitably recentered, *converges in a law to a limit which we describe.* We also provide a central limit theorem for the number of edges in a minimal-weight path.

We may see first passage percolation as defining a random metric on the graph, at least if the weights are positive. These limit theorems show that the distance between two vertices concentrates strongly around its mean value (i.e., the fluctuations are of order 1), and moreover provides a central limit theorem for the length of the geodesic.

The aim of this project is, first, to get accustomed to the tools used in [1], including: random walks, branching random walks, Chen-Stein method, first and second moment calculations, exploration processes of random graphs. Then, we aim to extend the results. Examples of possible extensions, in increasing complexity : more than two vertices, simple paths replaced by edge-disjoint paths, dense regime, different random graph (e.g. configuration model). This project may lead to the publication of a research article.

References

[1] Ma, H., & Maillard, P. (2025). Minimum-Weight Path in a Sparse Erdős—Rényi Graph with Signed Weights, arXiv:2511.22454, `https://doi.org/10.48550/arXiv.2511.22454`