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Super-linear spreading for Fisher-KPP equations set on antitrees

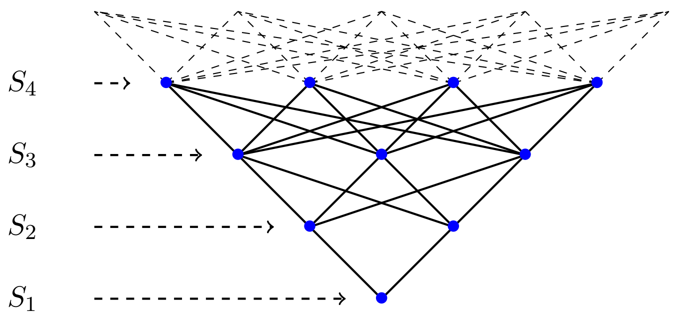
Internship summary. The overall objective of this M2 internship is to investigate the long time behavior of a class of differential equations defined on a particular class of graphs called antitrees. A connected simple rooted (infinite) graph is called an antitree if every vertex in the sphere S_n, which is the set of vertices of distance $n-1$ from the root, is connected to all vertices in the spheres S_{n-1} and S_{n+1} and no vertices in S_k for all $|k-n| \neq 1$. The class of differential equations considered in this internship belongs to the class of discrete reaction-diffusion equations, where the diffusion part is obtained by considering the graph Laplacian on an antitree, and the reaction part will be fixed to be logistic. More precisely, we shall consider equations of the form

$$\begin{cases} u_1'(t) = d_2(u_2(t) - u_1(t)) + \alpha_1 u_1(t)(1 - u_1(t)), \\ u_n'(t) = d_{n+1}u_{n+1}(t) - (d_{n+1} + d_{n-1})u_n(t) + d_{n-1}u_{n-1}(t) + \alpha_n u_n(t)(1 - u_n(t)), \quad \forall n \geq 2, \end{cases} \tag{1}$$

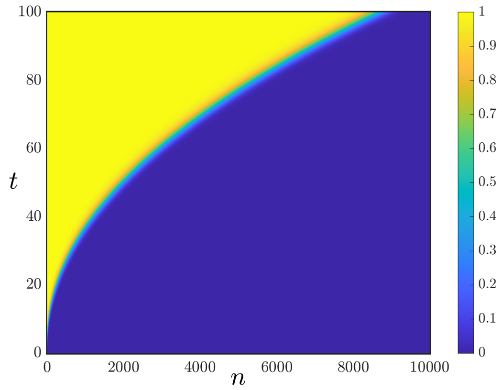
where $u_n(t)$ for $n \geq 1$ is a real unknown. Here, the sequence $(d_n)_{n\geq 1}$ is related to the nature of the antitree by the relation $d_n = |S_n|$ and $(\alpha_n)_{n\geq 1}$ is a given sequence, which we may assume for simplicity to be identically constant at the first place with $\alpha_n = \alpha > 0$ for all $n \geq 1$. Such types of systems naturally appear in the modeling of population dynamics, for example in ecology, epidemiology or neuroscience. In order to fix ideas, we shall work under the assumption that $d_n = \lfloor n^\gamma \rfloor$ for $\gamma > 0$ where $\lfloor \cdot \rfloor$ stands for the greatest integer part of a positive real number. We refer to Figure 1(a) for an illustration in the case $\gamma = 1$. Associated to (1), we shall consider compactly supported initial conditions, and for simplicity we can assume that

$$u_1(t=0) = 1, \quad \text{and} \quad \forall n \geq 2, \quad u_n(t=0) = 0. \tag{2}$$

The aim is this internship is thus to study the long time dynamics of (1)-(2) as a function of the parameter $\gamma > 0$. It is expected that super-linear propagation shall occur (see Figure 1(b) for an



(a) Antitree with sphere numbers $d_n = |S_n| = n$.



(b) Space-time plot of the solution $u_n(t)$ of (1)-(2).

FIGURE 1

illustration in the case $\gamma = 1$) and the objective will be to quantify this super-linear spreading as a function of the parameter $\gamma > 0$.

Prerequisites : PDE basics in particular for parabolic and elliptic equations, interests for graphs theory and mathematical biology, numerical simulations.

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Location : the internship will take place at the Institut de Mathématiques de Toulouse.

Dates : 5 months from March to July 2025

NB : an internship stipend is possible and, if successful, this internship may lead to a PhD thesis.