M2RI Internship Project

Hyperbolic systems with stabilizing and destabilizing mechanisms

Supervision:

— **Timothée Crin-Barat(100%)**
Associate Professor – IMT – Université de Toulouse
timothee.crin-barat@math.univ-toulouse.fr
timotheecrinbarat.com

Internship location: Institut de Mathématiques de Toulouse, Université de Toulouse

Language: English or French

Duration: March-July 2026

Possible continuation in PhD: Yes, applications for PhD positions will be strongly supported.

Application: CV, academic transcript, and list of M2 courses.

Prerequisites: Master 2 in Partial Differential Equations or Analysis.

Context: Nonlinear hyperbolic systems are known to form singularities (shocks) in finite time, even for small and smooth initial data, as shown in the classical works of Lax [5] and Serre [6]. In this project, we propose to investigate hyperbolic systems with dissipative and anti-dissipative terms, and to examine how these affect the behavior of the solutions: does the blow-up still occur? If so, is the blow-up time delayed compared to the case without dissipation? If not, how do the solutions behave for large times?

1 Influence of damping on the formation of singularities in hyperbolic systems

The first objective of this project is to understand how *weak solutions* behave under the influence of dissipative mechanism (such as friction) in the context of hyperbolic systems. In the case of the damped Burgers equation, it is known that damping modifies the blow-up time compared to the undamped dynamics, see [2, p.78]. Here, as a first step, we wish to review the result obtained in the literature concerning the damped p-system:

$$\begin{cases} \partial_t \rho - \partial_x u = 0, \\ \partial_t u + \partial_x P + u = 0, \\ (\rho, u)\big|_{t=0} = (\rho_0, u_0), \end{cases} \tag{1}$$

where, for $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $\rho := \rho(x, t)$ represents the density of the fluid, $u := u(x, t)$ the velocity and $P := P(\rho)$ the pressure. In particular, we shall investigate the so-called *diffusion phenomena* which allows the existence of global-in-time BV solution, see [3]. This result will also be compared with the findings of Sideris et al. [8], where a blow-up is observed for the three-dimensional compressible Euler system with damping.

Then, we shall generalize these analysis to the class of one-dimensional partially dissipative hyperbolic systems of the form

$$\partial_t U + A(U)\partial_x U + BU = 0 \tag{2}$$

where, for $n \in \mathbb{N}^*$, $U := U(t,x) \in \mathbb{R}^n$ is the unknown, A is a smooth symmetric matrix-valued function and B is a symmetric matrix such that

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$$

where D is a symmetric positive definite $n_2 \times n_2$ matrix with $n = n_1 + n_2$. The goal is to understand the interplay between the nonlinear term $A(U)$ and the linear dissipation D in the context of the blow-up time of the solutions.

2 Dissipative and anti-dissipative effects in a perturbative framework

In this section, we focus on strong solutions to hyperbolic systems that exist for all times. For hyperbolic systems with *sufficient dissipative* mechanisms, it is well-known that small and smooth initial data lead to global-in-time solutions. In particular, several works have shown that dissipation does not need to act on every equation of the system: in many cases, damping acting only some of the components suffices to stabilize the full dynamics. This phenomenon is related to the theory of hypocoercivity developed by Villani [9], see also the works of Shizuta and Kawashima [4, 7] and of Beauchard and Zuazua [1].

When considering small initial data, it is the linearization of the nonlinear dynamics around an equilibrium that dictates the behavior of the solutions. Therefore, we restrict ourselves to the analysis of linear one-dimensional systems of the form

$$\partial_t U + A\,\partial_x U + BU - CU = 0, \tag{3}$$

where, for $n \in \mathbb{N}^*$, $U := U(t,x) \in \mathbb{R}^n$, A is a symmetric $n \times n$ matrix, and B and C are positive definite $n \times n$ matrices such that

$$B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$$

where D is a positive definite $n_2 \times n_2$ matrix and G is a positive definite $n_1 \times n_1$ matrix with $n = n_1 + n_2$.

The objectives are the following:

- **Case $C = 0$.** When A and B satisfy the Kalman rank condition (see [1]), we aim to recover, through energy estimates, the sharp time-decaying estimates predicted by the spectral analysis. Classical hypocoercivity tools do not yield sharp decay rates, and our goal is to design a method that refines/optimizes the hypocoercive approach by splitting (potentially a very large number of times) appropriately the frequency domain.

- **Case $C \neq 0$.** Extend the results of Beauchard and Zuazua [1] by incorporating anti-damping effects into the system. The aim is to improve the understanding of the interplay between the conservative part A, the dissipative part B and the anti-dissipative terms C, and to identify sharp algebraic conditions on these matrices ensuring the L^2-stability of the system.

References

[1] K. Beauchard and E. Zuazua. Large time asymptotics for partially dissipative hyperbolic systems. *Arch. Rational Mech. Anal*, 199, 177–227, 2011.

[2] C.M. Dafermos. *Hyperbolic Conservation Laws in Continuum Physics*, volume 325 of *Grundlehren der mathematischen Wissenschaften*. Springer, New York, third edition, 2010.

[3] C.M. Dafermos and R. Pan. Global solutions for the p-system with frictional damping. *SIAM Journal on Mathematical Analysis*, 41, 2009.

[4] S. Kawashima. Systems of a hyperbolic-parabolic composite type, with applications to the equations of magnetohydrodynamics. *Doctoral Thesis*, 1983.

[5] P. D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves*, volume No. 11 of *Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1973.

[6] D. Serre. *Systèmes de lois de conservation, tome 1*. Diderot editeur, Arts et Sciences, Paris, New-York, Amsterdam, 1996.

[7] S. Shizuta and S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14, 249-275, 1985.

[8] T. Sideris, B. Thomases, and D. Wang. Long time behavior of solutions to the 3d compressible Euler equations with damping. *Comm. Partial Differential Equations*, 28, 795-816, 2003.

[9] C. Villani. Hypocoercivity. *Mem. Am. Math. Soc.*, 2010.