
Basic course A6 An introduction to the theoretical and numerical analysis of scalar conservation laws Examen

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The goal of this course is to introduce the audience to the study of partial differential equations of hyperbolic type and more precisely of scalar conservation laws of the form

$$\partial_t \rho(t, x) + \operatorname{div} F(t, x, \rho(t, x)) = 0,$$

for $t > 0$ and $x \in \Omega$ where Ω is an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) and where $F : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ (with $n \in \mathbb{N}^*$) is a given vector field function.

We will particularly concentrate on the linear case corresponding to $F(t, x, \rho) = \rho v(t, x)$ with $v : \mathbb{R} \times \Omega \rightarrow \mathbb{R}^n$ and on the one-dimensional (1-D) nonlinear case where $F(t, x, \rho) = f(\rho)$ with $f : \mathbb{R} \rightarrow \mathbb{R}$.

Prerequisites: Differential calculus and differential equations. Basic functional analysis. Lebesgue integration theory.

1. Modelling of transport phenomena

- Some examples in the 1-D case (traffic flow, gas dynamics, ...)
- The multi-D case : Reynolds transport theorem. Continuity and transport equations

2. The linear transport equation

- Smooth solutions by the method of characteristics
- Weak solutions of the linear transport equation. Jump conditions. Existence and uniqueness
- Finite Difference and Finite volume schemes in 1-D : construction, consistency, stability, convergence.
- Extensions to the multi-D setting.

3. Nonlinear conservation laws in 1-D

- Using the characteristics : local in time well-posedness of smooth solutions, finite time singularities, nonlinear regularization process
- Study of weak solutions : Rankine-Hugoniot conditions, global in time existence, non uniqueness
- Entropy solutions : physical motivations, definition and characterizations (Lax conditions, Oleinik conditions),
- Krushkov entropies. Uniqueness of the entropy solution by the doubling variables technique. Krushkov theorem.
- Riemann problems : definition, shocks and rarefaction waves

- Finite volume methods : monotone fluxes, TVD schemes. Convergence in 1-D towards an entropy solution for BV data

4. Possible extensions

- Few words concerning hyperbolic systems
- Finite volume schemes for scalar conservation laws in any dimension
- Boundary conditions

References

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