Speed of the N-branching Brownian motion

Stage M2RI

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Branching Brownian motion (BBM) is a branching process in which a population of particles randomly reproduces and moves on the real line. It is defined as follows. We start with one particle located at 0 at time 0. Each particle moves according to a Brownian motion during an exponentially distributed lifetime, then gives birth to a random number of children chosen according to a fixed reproduction law μ. These children appear at the position of their parent and then go on similarly independently of other particles. See Figure 1 for an illustration.

One can see the position of a particle as its biological fitness (or survival capacity) which fluctuates throughout its life due to mutations and is transmitted to its children. Then, in order to introduce natural selection in our model, we add a rule killing particles that are too low compared to the others. More precisely, we consider here a model called the N-BBM, introduced by [2] in physics and by [3] in mathematics, and defined for some fixed integer $N \geq 1$ as follows: particles evolves as in the BBM, but as soon as there are more than N particles, the lowest one is killed. See Figure 1.

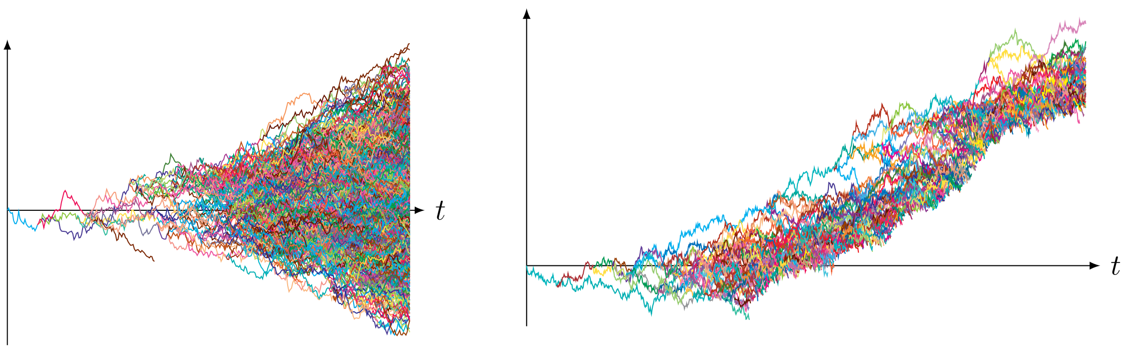

Figure 1: Realisation of a BBM on the left and a N-BBM (with $N = 50$) on the right.

An important question on this model is the study of the speed. If M_t denotes the maximal position at time t in the standard BBM, then we have

$$\frac{M_t}{t} \xrightarrow[t\to\infty]{\text{a.s.}} \sqrt{2}.$$

On the other hand, if M_t^N denotes the maximal position at time t in N-BBM, then we have

$$\frac{M_t^N}{t} \xrightarrow[t\to\infty]{\text{a.s.}} v_N,$$

for some deterministic $v_N \geq 0$. As the N-BBM is included in the standard BBM, it follows that $v_N \leq \sqrt{2}$. In order to quantify the effect of the selection on the speed for large N, one can show that

$$v_N = \sqrt{2} - \frac{\pi^2}{\sqrt{2}(\log N)^2} + o\left(\frac{1}{(\log N)^2}\right), \qquad \text{as } N \to \infty.$$

These facts have been proved in the case of the branching random walk, a discrete-time version of the BBM, in [1] and one then deduce the result for the N-BBM, see [4].

A first goal of this project would be to understand the methods of [1] and adapt them to the N-BBM case, in order to write down a direct proof in that case. Then, the second goal would be to investigate the behavior of the speed v_N when the reprodcution law μ has an infinite mean. In that case the previous results do not apply, the maximal speed for the BBM is infinite, so one can expect that $v_N \to \infty$, and the aim would be to find an asymptotic equivalent for v_N.

References

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