

Long-time behaviour for the Smoluchowski coagulation equations with source

Marina Ferreira

Institut de Mathématiques de Toulouse, CNRS, Université de Toulouse

marina.ferreira@math.univ-toulouse.fr

<https://marinaaferreira.com/>

M2RI internship, March - July 2026

Introduction The Smoluchowski coagulation equation governs the evolution of the size distribution of particles undergoing pairwise coalescence. It is one of the fundamental equations in the classical description of matter arising in statistical mechanics, alongside with the Boltzmann and the Vlasov equations. In its simplest form, the spatially homogeneous equation reads as follows

$$\partial_t f(x, t) = \mathbb{K}[f](x, t),$$

where $f(x, t)$ is the density of particles of size $x > 0$ at time $t \geq 0$. The coagulation operator is defined by

$$\mathbb{K}[f](x, t) := \frac{1}{2} \int_0^x K(x-y, y) f(x-y, t) f(y, t) dy - f(x, t) \int_0^\infty K(x, y) f(y, t) dy$$

where $K(x, y)$ represents the coagulation rate kernel at which two clusters of sizes x and y merge into a cluster of size $x + y$.

Since the coagulation operator is nonlocal, the solutions can have a rich, and sometimes counterintuitive, behaviour depending on the choice of the coagulation kernel. One example is the loss of particles in the system, which occurs for kernels that grow fast enough near infinity. This phenomenon is associated to the unboundedness of some integrals and it is nowadays widely understood as the gelling transition or *gelation*: the formation of an infinitely large particle (called gel) which does not interact with the finitely large ones, leading the total mass to decrease [1].

In spite of the presence or not of gelation, the long time behaviour is always expected to be *self-similar*, meaning that the solutions are expected to approach a self-similar function f_s of the form

$$f_s(x, t) = \frac{1}{t^p} \phi\left(\frac{x}{t^q}\right),$$

where ϕ is called the self-similar profile.

Specific coagulation kernels have been proposed in a wide range of applications, including aerosol growth, polymerization, formation of planets, and also social networks and

animal grouping. Besides coagulation, other processes can also be found in the literature, such as fragmentation, growth, source and sink.

Goals In this project, we will focus on the study of the Smoluchowski coagulation equation with an additional *source term*,

$$\partial_t f(x, t) = \mathbb{K}[f](x, t) + \eta(x),$$

that has been mostly absent from the mathematical literature. We will consider a general class of coagulation kernels, including those encountered in applications. The aim is to study the long-time behaviour that has been recently conjectured in [2] and in [3]. The mere addition of particles in the system, can change its qualitative long-time behaviour completely, leading to the existence of non-trivial steady states [4], "anomalous" self-similar scalings and degenerate self-similar profiles [3]. This project aims at exploring these long time effects numerically, as well as analytically in the case of simple kernels for which an analytical solution is available.

In fact, since the equation is non-local, numerical methods have to rely on *apriori* knowledge of the solution to ensure that no mass is being lost along the simulation due to the numerical truncation of the infinite sums. (Some numerical methods and simulations have been developed in [5, 6].) Therefore, analytical and numerical results will be developed side by side so to achieve a correct understanding of the long-time behaviour. Analytical tools will range from formal methods to obtain the scalings and the asymptotics to rigorous methods based on measure theory and Laplace transforms for explicitly solvable kernels, that have been previously developed in the case without source.

Possible topics to approach are the following:

- Analysis of the coagulation equation with source and constant kernel, $K(x, y) = 1$, using Laplace transforms.
- Implementation of several algorithms for integro-differential equations and numerical study in the case of the constant kernel.
- Numerical and analytical investigation of the different scenarios conjectured in [2] and [3] for general kernels.
- Exploration of the gelling kernels and formulation of new conjectures about the long time behaviour in this regime.

Perspectives for a PhD thesis Despite the progress over the last century on the analysis of the Smoluchowski coagulation equation, the precise long time behaviour remains a central open problem for most coagulation kernels, in particular, for those used in applications. The methods developed for linear nonlocal equations, such as standard semi-group theory or the Harry's theorem that has been used to characterize precisely the long time asymptotics of growth-fragmentation equations, do not apply [1]. The main difficulty is the lack of uniqueness of self-similar profiles. Nevertheless, it turns out that in the case of the coagulation equation with source and zero initial data, there is one time-dependent solution that is exactly a self-similar solution [2]. This solution is expected to

be unique, which would imply uniqueness of the self-similar profile. Opening the route towards obtaining the precise long time behaviour for general kernels and general initial condition in the case of the coagulation equation with source. This could be done as part of a PhD project.

Tools and background Strong background in analysis and numerical analysis, in particular, in the analysis and simulation of PDEs.

References

- [1] Banasiak, J., Lamb, W., Laurençot, P. (2019). *Analytic Methods for Coagulation-Fragmentation Models, Volume I*. Chapman and Hall/CRC.
- [2] Ferreira, M. A., Franco, E., Velázquez, J. J. (2022). On the self-similar behavior of coagulation systems with injection. *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, 40(4), 803-861.
- [3] Ferreira, M. A., Franco, E., Lukkarinen, J., Nota, A., Velázquez, J. J. (2023). Coagulation equations with source leading to anomalous self-similarity. *Journal of Physics A: Mathematical and Theoretical*, 56(48), 485002.
- [4] Ferreira, M. A., Lukkarinen, J., Nota, A., Velázquez, J. J. (2021). Stationary non-equilibrium solutions for coagulation systems. *Archive for rational mechanics and analysis*, 240(2), 809-875.
- [5] Filbet, F., Laurençot, F. "Numerical simulation of the Smoluchowski coagulation equation." *SIAM Journal on Scientific Computing* 25.6 (2004): 2004-2028.
- [6] Davies, S. C., King, J. R., Wattis, J.. "The Smoluchowski coagulation equations with continuous injection." *Journal of Physics A: Mathematical and General* 32.44 (1999): 7745.