**Master level internship**

## Limit theorems for adaptative quantum measurements

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Repeated quantum measurements describe contemporary experiments in quantum technologies, in particular in quantum optics. From a mathematics point of view, the formalism of quantum mechanics is a generalization of the classical probability to non commutative objects. Typically probability vectors are replaced by positive semi-definite matrices of trace one (density matrices) and (sub-)stochastic matrices are replaced by positive maps on positive semi-definite matrices.

Many standard results on Markov chains can be generalized to repeated quantum measurements (Perron-Frobenius theorem, law of large numbers, central limit theorem ...). However, most of these results are limited to independent copies of the same measurement instrument. Leveraging the memory of the past measurement outcomes to engineer more subtle measurements protocols is gaining interest in the physics community. These are called adaptative measurements. In this internship, we propose to the interested student to study the mathematics of adaptative measurements with the goal of proving some limit theorems for the measurement outcomes.

Mathematically, quantum measurements are modeled by families of positive endomorphisms $\{\Phi_a\}_{a\in\mathcal{A}}$ on positive semi-definite matrices such that their sum $\Phi = \sum_a \Phi_a$ preserves the trace : $\mathrm{tr}(\Phi(A)) = \mathrm{tr}(A)$. For this internship we will consider bi-infinite sequences of measurement outcomes: $\Omega = \mathcal{A}^{\mathbb{Z}}$. Then, to a past $\omega_- \in \mathcal{A}^{\mathbb{Z}_-}$ (*i.e.* $\omega_- = (\ldots, \omega_{-1}, \omega_0)$), corresponds an instrument $\{\Phi_{\omega_-:a}\}_{a\in\mathcal{A}}$ such that $\Phi_{\omega_-} = \sum_a \Phi_{\omega_-:a}$ preserves the trace. Then, the goal of the internship would be to study, if it exists, the dynamical system $(\Omega, \varsigma, \mathbb{P})$ with $\varsigma : \Omega \to \Omega, \varsigma(\omega)_n = \omega_{n+1}$ and $\mathbb{P}$ describing the law of the measurement outcomes according to quantum mechanics.

The first step in the internship would be to prove the existence (and eventual uniqueness) of $\mathbb{P}$. Towards that goal some standard techniques related to Ruelle's transfer operators – see Ruelle's Perron-Frobenius Theorem in [BC] – can be adapted to the non commutative setting. Then, under appropriate assumptions, using the spectral properties of this transfer operator, some limit theorems (Central limit theorem, large deviation principle ...) could be established using extensions of standard techniques in ergodic theory.

This internship could be pursued in a Ph. D. thesis whose goal would be to relate the measure $\mathbb{P}$ to an appropriate variational principle. Indeed, repeated quantum measurements have been proven to be quite singular models of spins chains in classical statistical mechanics – see [BCJP] and references therein. While not being Gibbs, the measure $\mathbb{P}$ (when $\Phi_{\omega_-:a}$ does not depend on $\omega_-$) verify a thermodynamical variational principle with respect to a very singular and long range potential. Extending such a variational principle by including a dependency on the past could unlock the study of phases of matter for new interesting spin chain models with long range interaction that may relate to questions in other fields of mathematics such as number theory.

# References

[BC] Bowen, Rufus, and Jean-René Chazottes. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Vol. 470. Berlin: Springer, 1975.

[BCJP] Benoist, Tristan, Noé Cuneo, Vojkan Jakšić, and Claude-Alain Pillet. "On entropy production of repeated quantum measurements II. Examples." *Journal of Statistical Physics* 182, no. 3 (2021): 44.