Statistical Analysis of a Surface Gravity Wave Model: Weak Wave Turbulence, Soliton Gas, and Breaking-Induced Turbulence

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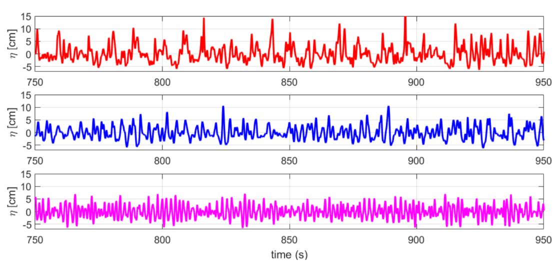
Context

The mathematical modeling of dispersive nonlinear waves plays a central role in oceanography, fluid dynamics, and applied mathematics. Depending on the strength of dispersion and nonlinearity, wave fields exhibit markedly different statistical behaviors, including:

- weak wave turbulence, governed by quasi-Gaussian resonant interactions;
- soliton gas dynamics, arising in weakly dispersive nonlinear regimes where coherent structures dominate;
- breaking-induced turbulence, associated with wave breaking



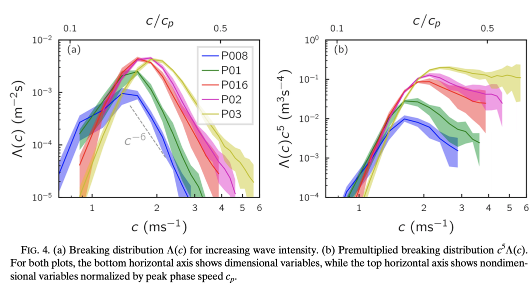


Figure 1: On the left: from bottom figure to top figure, the emergence of solitons and their interactions from a random data. [Leduque et al, 2025]. On the right: Breaking distribution for the free surface Navier-Stokes equations (with 15 layers of discretization in the vertical direction) [Wu et al, 2025].

The goal of this internship is to carry out a theoretical and numerical investigation of a nonlinear dispersive wave model, so called Leucothea (LCT), which is an hyperbolic version of the well-known Serre-Green-Naghdi equations. More specifically, we aim at characterizing the statistical properties of the solutions and identify the transition between the three dynamical regimes for the LCT model and compare it to experimental data or numerical ones on the direction numerical simulation of free surface Euler equations.

Objectives

Theoretical Study

The first part of the internship will consist in a bibliographic study with an introduction to the various water models found in the literature like NLS or KdV and more evolved ones like Boussinesq, Serre-Green-Naghdi and Leucothea and their connections. The student will also study the main features of these models, in particular study its invariants (mass, energy, momentum) and relevant asymptotic scalings. Then, a review the key concepts will be of interest:

- kinetic theory of weak wave turbulence;
- statistical theory of soliton gases (Zakharov–Gelash type distributions, random soliton ensembles);
- turbulence in wave breaking.

Numerical Study

The second part of the internship is dedicated to a numerical study: it will be carried out with TOLOSA, the open source software developed at INSA and used in SHOM numerical environment. The student will generate numerical datasets for various initial conditions (Gaussian random fields, localized pulses, filtered noise, etc.) and perform a systematic statistical analysis: amplitude, phase, and frequency distributions and compare the observed statistics with theoretical predictions (Kolmogorov–Zakharov spectra, Poissonian soliton statistics, heavy-tailed laws in breaking regimes).

Expected Results

We expect a phase diagram of the dynamical regimes as functions of the governing parameters (nonlinearity, dispersion) so as to identify transitions from weak turbulence to soliton gas and soliton gas to breaking-induced turbulence. This will help to determine the validity and limitations of asymptotic theories in strongly nonlinear regimes.

Required Skills

- Solid background in PDEs and nonlinear dispersive models.
- Proficiency in scientific computing
- Interest in probability, statistics, and signal analysis.

Skills Developed During the Internship

- Implementation of numerical simulations of nonlinear wave fields.

- Understanding of turbulence, kinetic theory, and soliton dynamics.
- Statistical analysis of high-dimensional and non-Gaussian data.
- Scientific writing connecting theoretical predictions with numerical observations.

This internship comes with a gratification from SHOM. It will take place at IMT (Institut de Mathématiques de Toulouse) in the Applied Mathematics Department of INSA Toulouse. There is a possibility to continue this internship with a PhD thesis.

Bibliography

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Soliton Gas / Integrable Turbulence

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