Master level internship

Purification of quantum trajectories in infinite dimension

Tristan Benoist: tristan.benoist@math.univ-toulouse.fr
Clément Pellegrini: clement.pellegrini@math.univ-toulouse.fr

Repeated quantum measurements describe contemporary experiments in quantum technologies, in particular in quantum optics. From a mathematics point of view, the evolution of a quantum system subject to repeated measurements is described by a Markov chain on density operators (*i.e.* positive semi-definite trace-class operators of trace 1) on a complex Hilbert space $\mathcal{H}$. These Markov chains are called quantum trajectories. The chain is defined using a measurable function V from a measurable set $\mathcal{A}$ to bounded operators on $\mathcal{H}$ (*i.e.* $V : \mathcal{A} \ni a \mapsto V_a \in \mathcal{B}(\mathcal{H})$) such that there exists a measure μ over $\mathcal{A}$ such that $\int_{\mathcal{A}} V_a^* V_a d\mu(a) = \mathrm{Id}_{\mathcal{H}}$. The update rule for the chain is the following: given that the chain is in state ρ_n at time n, the state of the chain at time $n+1$ is,

$$\rho_{n+1} = \frac{V_A \rho_n V_A^*}{\mathrm{tr}(V_A^* V_A \rho_n)} \quad \text{with } A \text{ distributed according to } \mathrm{tr}(V_a^* V_a \rho_n) d\mu(a).$$

The set where this Markov chain takes value, the set of density operators, denoted $\mathcal{D}(\mathcal{H})$, is convex and, when $\dim \mathcal{H} < \infty$, compact. In that latter case it has been proved that $(\rho_n)_n$ has a tendency to get close to the set of extreme points of $\mathcal{D}(\mathcal{H})$ – see [KM]. These extreme points are called pure states. Thus this phenomenon has been dubbed purification.

Purification is at the heart of the study of quantum trajectories. It is, for example, instrumental in the proof of the uniqueness of the invariant measure for $(\rho_n)_n$ – see [BFPP] – and subsequent limit theorems – see [BFP, BHP] – or to show convergence to pointer states for non demolition measurements – see [BBB].

The study of purification when $\dim \mathcal{H}$ is infinite is in its infancy. In that case $\mathcal{D}(\mathcal{H})$ is no-longer compact so one has to find a new compactness argument. It has been proved only for a handful of models – see [BBFF, BBP] – and general results – see *e.g.* [GV] – are not satisfactory yet. In this internship, we propose to the interested student to study purification in some models of quantum trajectories in infinite dimension. He or she would work on the convergence of non demolition measurements for mean-field non-linear models or on general Gaussian measurement models.

This internship could be pursued into a Ph. D. thesis whose goal would be to establish purification for more general quantum trajectories models and leverage these results to prove uniqueness of the invariant measure for those models.

References

[BBB] M. Bauer, T. Benoist, D. Bernard, *Repeated Quantum Non-Demolition Measurements: Convergence and Continuous Time Limit*, Ann. Henri Poincaré **14** (2013) 639 – 679

[BBFF] M. Ballesteros, T. Benoist, M. Fraas, and J. Fröhlich, *The appearance of particle tracks in detectors*, Commun. Math. Phys. **385** (2021) 429 – 463

[BBP] T. Benoist, L. Bruneau and C. Pellegrini, *Quantum trajectory of the one atom maser*, Probab. Math. Phys. **6**(3) (2025) 1073 – 1110

[BFP] T. Benoist, J.-L. Fatras and C. Pellegrini, *Limit theorems for quantum trajectories*, Stoch. Process. Their Appl. **164** (2023) 288 – 310

[BFPP] T. Benoist, M. Fraas, Y. Pautrat and C. Pellegrini, *Invariant measures for quantum trajectories*, Probab. Theory Relat. Fields **174** (2019) 307 – 334

[BHP] T. Benoist, A. Hautecœur and C. Pellegrini, *Spectral gap and limit theorems for quantum trajectories*, J. Funct. Anal. **289.5** (2025) 110932

[GV] F. Girotti and A. Vitale, *Purification of quantum trajectories in infinite dimensions*, arXiv preprint arXiv:2509.13377 (2025)

[KM] H. Maassen and B. Kümmerer, *Purification of quantum trajectories*, Lecture Notes-Monograph Series (2006) 252 – 261.