Master level internship

Perturbation theory for quantum trajectories

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Repeated quantum measurements describe contemporary experiments in quantum technologies, in particular in quantum optics. From a mathematics point of view, the evolution of a quantum system subject to repeated measurements is described by a Markov chain on the projective space of $\mathbb{C}^d$ (denoted $P(\mathbb{C}^d)$). These Markov chains are called quantum trajectories. The chain is defined using a finite set of $d \times d$ matrices $\{V_i\}_i$ such that $\sum_i V_i^* V_i = \mathrm{Id}_d$. The update rule for the chain is the following: given that the chain is in state $\hat{x}_n$ at time n, the state of the chain at time $n+1$ is,

$$\hat{x}_{n+1} = \widehat{V_i x} \quad \text{with probability } \|V_i x\|^2$$

with x a unit norm vector representative of $\hat{x}$ and $\widehat{V_i x} \in P(\mathbb{C}^d)$ is the equivalent class of $V_i x$ in $P(\mathbb{C}^d)$. The associated Markov kernel is given, for any bounded measurable function f, by

$$\Pi f(\hat{x}) = \sum_i f(\widehat{V_i x}) \|V_i x\|^2.$$

These processes are quite singular and the usual tools to study Markov chains such as φ-irreducibility are not adapted to them. Recently, the first proof of uniqueness of the invariant measure for quantum trajectories under standard assumptions has been established – see [BFPP]. Following this result different techniques have been used to establish limit theorems – see [BFP, BHP]. In particular, in [BHP], a spectral gap property has been proved.

The goal of this internship is to leverage these results to study the perturbation theory for these Markov chains. This would allow some perturbation computation of more complex models of quantum trajectories based on well understood ones. It would also allow for the application of results based on perturbation theory such as adiabatic theorems or linear response theory.

By perturbation theory we mean establishing some regularity (smoothness, analycity ...) of Π when the matrices $\{V_i\}_i$ vary. During this internship, the student will first familiarize his or herself with the mathematical setting of [BHP]. Then some proof of regularity of Π for some toy models of matrices $\{V_i\}_i$ will be explored. This internship can be pursued into a Ph. D. during which the full perturbation theory could be developed along with some relevant consequences.

Beyond quantum trajectories, we expect this project to shed light on the properties of random products of matrices such as in [Haut].

References

[BFPP] T. Benoist, M. Fraas, Y. Pautrat and C. Pellegrini, *Invariant measures for quantum trajectories*, Probab. Theory Relat. Fields **174** (2019) 307 - 334

[BFP] T. Benoist, J.-L. Fatras and C. Pellegrini, *Limit theorems for quantum trajectories*, Stoch. Process. Their Appl. **164** (2023) 288 - 310

[BHP] T. Benoist, A. Hautecœur and C. Pellegrini, *Spectral gap and limit theorems for quantum trajectories*, J. Funct. Anal. **289.5** (2025) 110932

[Haut] A. Hautecoeur, *Analyticity of the pressure function for products of matrices*, arXiv preprint arXiv:2501.03590 (2025).