Francis Filbet
Institut de Mathématiques de Toulouse
CNRS, UMR 5219
Université Paul Sabatier
Toulouse

Master internship and PhD in Mathematics

Subject: On a discrete framework of hypocoercivity for kinetic equations.

Summary.

Hypocoercivity is a concept that arises in the study of partial differential equations, particularly in the context of the long-time behavior of solutions to these equations. It refers to a class of systems that exhibit a combination of coercivity (which ensures the dissipation of energy) and hypoellipticity (which ensures that solutions become smooth over time).

In this internship program, we consider the simplest model the Vlasov-Poisson system, which describes the evolution of charged particles in the case where the only interaction considered is the mean-field force created through electrostatic effects. The system consists in Vlasov equations for phase space density $f(t, \mathbf{x}, \mathbf{v})$

$$\begin{cases} \dfrac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \mathbf{E} \cdot \nabla_{\mathbf{v}} f = Q(f)\,, \\[2ex] f(t=0) = f_0\,, \end{cases} \tag{1}$$

coupled to its self-consistent electric field $\mathbf{E} = -\nabla_{\mathbf{x}}\Phi$ which satisfies the Poisson equation

$$-\Delta_{\mathbf{x}}\Phi = \rho - \rho_0\,, \qquad \text{with} \quad \rho = \int_{\mathbb{R}^d} f \,\mathrm{d}\mathbf{v}\,, \tag{2}$$

where ρ_0 is such that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} f \,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{x} = \int_{\mathbb{T}^d} \rho_0 \,\mathrm{d}\mathbf{x}.$$

Here we consider that the collision operator $Q(f)$ is given by the Fokker-Planck operator

$$Q(f) = \nu \,\mathrm{div}_{\mathbf{v}} \left(\mathbf{v} f + \nabla_{\mathbf{v}} f\right),$$

where $\nu > 0$. In previous works [1, 2], using hypocoercive techniques, we studied the long time behavior of the solution when $t \to \infty$, where the solution converges to a unique equilibrium called the Maswellian distribution function

$$\mathcal{M}(v) = \frac{1}{(2\pi)^{d/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2}\right).$$

Furthermore, we proposed and analysed a numerical scheme for the Vlasov-Poisson-Fokker-Planck model. On the one hand, we proved that our method is asymptotic preserving in the long time regime for the linearized model. To do so, we derived the exponential relaxation of the numerical solution towards its equilibrium with rates independent of scaling and discretization parameters.

An important continuation of this work is to incorporate nonlinear collisions to the model. Let us first observe that in [3], a spectral method is applied to a nonlinear Fokker-Planck operator conserving mass, momentum and energy. However, extending our analysis of the longtime regime at the discrete level to this case may require modifications and further investigations have to be done. L^2-hypocoercivity methods have been applied in the case of nonlinear collision operators and linearized Boltzmann operators at the continuous level [5], however such analysis at the discrete level is not available in the literature in the framework of Hermite decomposition.

Another possible continuation would consist in finding a non-homogeneous background configuration where damping phenomena occur as in the homogeneous case, and then construct an experiment where nonlinear effect play the main role, even for small perturbation as for plasma echoes.

Prerequisites: Master 2 Research in Mathematics or Applied Mathematics.

Contact: francis.filbet@math.univ-toulouse.fr

References

[1] A. Blaustein and F. Filbet, *On a discrete framework of hypocoercivity for kinetic equations.* Math. Comp. 93 (2024), no. 345, 163-202.

[2] A. Blaustein and F. Filbet, *A structure and asymptotic preserving scheme for the Vlasov-Poisson-Fokker-Planck model.* J. Comput. Phys. 498 (2024), Paper No. 112693, 25 pp.

[3] F. Filbet and C. Negulescu, *Fokker-Planck multi-species equations in the adiabatic asymptotics*, J. Comput. Phys. 471 (2022), Paper No. 111642, 28 pp.

[4] François Golse. *On the Dynamics of Large Particle Systems in the Mean Field Limit.* Cours Master Ecole Polytechnique.

[5] F. Hérau, Introduction to Hypocoercive Methods and Applications for Simple Linear Inhomogeneous Kinetic Models, Lectures on the analysis of nonlinear partial differential equations. Part 5, Morningside Lect. Math., vol. 5, Int. Press, Somerville, MA, 2018, pp. 119-147.

[6] C. Villani, *Hypocoercivity*, Mem. Am. Math. Soc. (2009).