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Master internship and PhD in Mathematics

Subject: On a discrete framework of hypocoercivity for kinetic equations.

Summary.

Hypocoercivity is a concept that arises in the study of partial differential equations, particularly in the context of the long-time behavior of solutions to these equations. It refers to a class of systems that exhibit a combination of coercivity (which ensures the dissipation of energy) and hypoellipticity (which ensures that solutions become smooth over time).

In this internship program, we consider the Kuramoto-Sakaguchi system, which describes the evolution of particles in the case where the only interaction considered is the mean-field force created through particle interactions. The system consists in Vlasov equations for phase space density $f(t, \theta, \omega)$

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega \cdot \partial_\theta f + \mathbf{F}\, \partial_\omega f = Q(f)\,, \\ f(t=0) = f_0\,, \end{cases} \tag{1}$$

coupled to its self-consistent electric field $\mathbf{F} = -\sin \star \rho$ with

$$\rho = \int_{\mathbb{R}} f \, \mathrm{d}\omega\,, \tag{2}$$

Here we consider that the collision operator $Q(f)$ is given by the Fokker-Planck operator

$$Q(f) = \partial_\omega \left(\omega f + \partial_\omega f\right).$$

In previous works [1–3], using hypocoercive techniques, we studied the long time behavior of the solution when $t \to \infty$, where the solution converges to a unique equilibrium called the Maswellian distribution function

$$\mathcal{M}(\omega) = \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{|\omega|^2}{2}\right).$$

In this internship proposal, we want to propose and analyse a numerical scheme for the Kuramoto-Sakaguchi model. On the one hand, we will prove that our method is asymptotic preserving in the long time regime for the linearized model. To do so, we will derive the exponential relaxation of the numerical solution towards its equilibrium with rates independent of scaling and discretization parameters.

An important continuation of this work is to incorporate nonlinear collisions to the model. Spectral methods may be applied to a nonlinear Fokker-Planck operator conserving mass, momentum and energy. However, extending our analysis of the longtime regime at the discrete level to this case may require modifications and further investigations have to be done. L^2-hypocoercivity methods have been applied in the case of nonlinear collision operators and linearized Boltzmann operators at the continuous level [5], however such analysis at the discrete level is not available in the literature in the framework of Hermite decomposition.

Prerequisites: Master 2 Research in Mathematics or Applied Mathematics.

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References

[1] A. Blaustein and F. Filbet, *On a discrete framework of hypocoercivity for kinetic equations.* Math. Comp. 93 (2024), no. 345, 163-202.

[2] A. Blaustein and F. Filbet, *A structure and asymptotic preserving scheme for the Vlasov-Poisson-Fokker-Planck model.* J. Comput. Phys. 498 (2024), Paper No. 112693, 25 pp.

[3] F. Filbet and M. Kang, *Trend to equilibrium and diffusion limit for the inertial Kuramoto-Sakaguchi equation* , preprint 2023.

[4] F. Golse. *On the Dynamics of Large Particle Systems in the Mean Field Limit.* Cours Master Ecole Polytechnique.

[5] F. Hérau, Introduction to Hypocoercive Methods and Applications for Simple Linear Inhomogeneous Kinetic Models, Lectures on the analysis of nonlinear partial differential equations. Part 5, Morningside Lect. Math., vol. 5, Int. Press, Somerville, MA, 2018, pp. 119-147.

[6] C. Villani, *Hypocoercivity*, Mem. Am. Math. Soc. (2009).