Elliptic PDEs and Calculus of Variations

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The aim of this course is to present elementary methods used in the study of elliptic partial differential equations (PDE), with particular emphasis on the problems of existence, uniqueness and regularity of solutions, including connections with the Calculus of Variations.

0. Crash course:
Functional Analysis, Distributions, Sobolev spaces.

1. Introduction:
Elliptic operators of divergence form, Weak solutions.

2. Laplace's Equation:
Regularity, Quantitative properties of harmonic functions (Caccioppoli's inequality, H^k estimates, Compactness....)

3. Regularity of weak solutions:
De Giorgi's theorem ($C^{0,\alpha}$ regularity), Moser's iteration technique.

4. Boundary value problems:
Lax-Milgram's Theorem, Elliptic regularity for H^{-1} and L^2 data.

5. Maximum principle:
Weak Maximum principle, Hopf's lemma, Strong Maximum principle, Moving plane method, Symmetry of positive solutions.

6. Semi-linear elliptic PDEs:
Local invertibility, Nemitskii operators, Global invertibility (Continuity method), Nonexistence results (Pohozaev's identity).

7. Calculus of variations and critical points:
The Direct Method, Constrained Minimization, Euler-Lagrange equations, Morse's Deformation Theorem, Palais-Smale sequences, Mountain Pass lemma, Applications to semi-linear elliptic PDEs.

8. Compactness results:
Concentration-compactness, compensated-compactness. Applications to elliptic PDEs.

References:

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3. Gilbarg, D. and Trudinger, N. *Elliptic partial differential equations of second order.* Springer-Verlag, Berlin, 2001.
4. Giaquinta, M. *Introduction to regularity theory for nonlinear elliptic systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.
5. Struwe, M. *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems.* Springer-Verlag, Berlin 2008.