The spectrum of the Laplacian on hyperbolic surfaces

The aim of this Reading Seminar will be to study some of the properties of the eigenfunctions of the Laplacian on hyperbolic surfaces. These objects, analytic in nature, contain a lot of information on the geometry of the surfaces. One of the aims of this Reading Seminar is *Selberg's trace formula*, which establishes a deep relation between the lengths of the closed geodesics on a compact surface and the eigenvalues of the Laplacian. We will give some of its applications, in particular to the counting problem of the closed geodesics on a hyperbolic surface. The error term in this estimate involves the "small eigenvalues", *i.e.*, the eigenvalues in the interval $(0, \frac{1}{4})$. We could end with a theorem by Sarnak and Xue: for every prime number p large enough, the quotient of the upper half plane by the congruence subgroup $\Gamma(p)$ does not contain any eigenvalues lower than $\frac{5}{36}$.

Hyperbolic Geometry [Ber16, 1], [Bus10, 1,2], [Kat92, 1,2]. Models for the hyperbolic plane: the disk, the upper half plane. Description of the geodesics. Isometries. Computation of the volume of polygons.

Fuchsian groups [Ber16, 1], [Bus10, 3], [Kat92, 2, 3]. These are the discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$. Hyperbolic surfaces. Constructions of hyperbolic surfaces, compact or of finite volume, by gluings of polygons, and gluings of hyperbolic pairs of pants.

Arithmetic Fuchsian groups [Ber16, 2], [Kat92, 3,4] Quaternion algebras. Construction of the Fuchsian groups $\Gamma_{a,b}$ out of quaternion algebras. Congruence groups, the congruence group $\Gamma(N) = \ker(\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}))$. Topology of the congruence surfaces $\mathbb{H}/\Gamma(N)$.

The Laplacian [Ber16, 3], [Cha84, 1] Definition of the Laplacian acting on functions. Examples of proper functions in the disk model, and in the upper half plane model. Green's formulas.

Spectral theory of the Laplacian on compact surfaces [Ber16, 3], [Bus10, 7]. Study of the Laplacian as a self-adjoint operator defined on a dense subset of $L^2(\mathbb{H}/\Gamma)$. Spectral theorem: there exists an orthonormal basis on $L^2(\mathbb{H}/\Gamma)$ consisting of eigenfunctions.

Spectral theory of the Laplacian on $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ [Ber16, 3]. When Γ has finite covolume, construction of the Eisenstein series $E(z, s)$. When $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, meromorphic continuation of the Eisenstein series. Spectral theorem for the Laplacian on $L^2(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z}))$. Existence of infinitely many linearly independent eigenfunctions on $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ which vanish at infinity.

The trace formula and applications [Ber16, 5], [Bus10, 9]. Pretrace formulas, trace formulas when Γ is cocompact: these are relations between two series, one involving the eigenvalues of the Laplacian, the other involving the lengths of the closed geodesics on $\mathbb{H}/\Gamma$. Counting of the number of closed geodesics of length $\leqslant R$ and the error term which involves the small eigenvalues.

Geometric criteria for the existence of small eigenvalues [Ber16, 3], [Bus10, 8], [Cha84, 1]. Variational characterization of the small eigenvalues of the Laplacian. Construction of surfaces of genus g with small eigenvalues.

A theorem of Sarnak-Xue [Ber16, 6] For every prime number p large enough, the surface $\mathbb{H}/\Gamma(p)$ has no eigenvalues less than $\frac{5}{36}$.

Additional references may include [Hel81, Iwa95].

References

[Ber16] Nicolas Bergeron. *The spectrum of hyperbolic surfaces. Translated from the French by Farrell Brumley.* Universitext. Les Ulis: EDP Sciences; Cham: Springer, 2016.

[Bus10] Peter Buser. *Geometry and spectra of compact Riemann surfaces.* Mod. Birkhäuser Class. Boston, MA: Birkhäuser, reprint of the 1992 original edition, 2010.

[Cha84] Isaac Chavel. *Eigenvalues in Riemannian geometry. With a chapter by Burton Randol. With an appendix by Jozef Dodziuk*, volume 115 of *Pure Appl. Math., Academic Press.* Academic Press, New York, NY, 1984.

[Hel81] Sigurdur Helgason. *Topics in harmonic analysis on homogeneous spaces*, volume 13 of *Prog. Math.* Birkhäuser, Cham, 1981.

[Iwa95] Henryk Iwaniec. *Introduction to the spectral theory of automorphic forms.* Madrid: Biblioteca de la Revista Matemática Iberoamericana, 1995.

[Kat92] Svetlana Katok. *Fuchsian groups.* Chicago: The University of Chicago Press, 1992.